International Journal of Theoretical Physics, Vol. 27, No. 2, 1988

Axially Symmetric Solution to Rosen's Field Equations with Angular Momentum

R. J. Knill,[1] W. R. Stoeger,[2] and A. P. Whitman[3]

Received August 18, 1987

There has been no Kerr-like solution to Rosen's bimetric theory of gravity, in the sense that there is no stationary, axially symmetric solution with angular momentum term. Here such a solution is derived and investigated.

1. INTRODUCTION

Rosen's bimetric theory of gravitation was recognized by Stoeger (1983) as a harmonic map. Whitman *et al.* (1986) exploited this fact to obtain four broad classes of solutions to Rosen's field equations. Neither these nor Rosen's own (static spherically symmetric) solution are Kerr-like in the sense that they are axially symmetric stationary solutions with an angular momentum term. It is the purpose of this paper to provide an example of one such solution and to suggest at least a preliminary physical interpretation of it.

It will be useful to briefly review how the four classes of solutions were obtained in Whitman *et al.* (1986). Briefly, then, suppose that $\mathbb{R}^4$ is Minkowski space-time. While Rosen's equations are covariant, it simplifies matters to take on $\mathbb{R}^4$ the canonical coordinates t, x, y, z and to identify a Lorentzian metric on $\mathbb{R}^4$ as a map (with singularities) F of $\mathbb{R}^4$ into the space $\mathcal{M} = 4\times 4$ symmetric matrices equipped with the Dewitt metric $\mathcal{G}$. By Stoeger (1983), a Lorentzian metric satisfies Rosen's field equations if and only if F is a harmonic map in the sense of Eells and Sampson (1964), generalized *mutatis mutandis* to the case of semi-Riemannian manifolds.

[1]Department of Mathematics, Tulane University, New Orleans, Louisiana 70118.
[2]Specola Vaticana, I-00120, Vatican City, Italy.
[3]Department of Mathematics, Loyola University of the South, New Orleans, Louisiana 70118, and Pontificia Universidade Catolica do Rio de Janeiro, 22.451 Rio de Janeiro, Brazil.

0020-7748/88/0200-0283$06.00/0

Although the word "harmonic" should conjure up elliptic equations, our generalized notion of harmonic map is in fact a solution to hyperbolic equations, with all that that entails. However, it is fortunate that one aspect of the theory of Eells and Sampson carries over to our case. That is the fact that a map $F:\mathbb{R}^4\to\mathcal{M}$ is "harmonic" if it factors into two maps, $F=S\circ\psi$, where $\psi:\mathbb{R}^4\to\mathbb{R}^n$ and $S:\mathbb{R}^n\to\mathcal{M}$ are such that ψ is harmonic with respect to the Euclidean metric $|\circ|$ on $\mathbb{R}^n$, and S is totally geodesic. This is a real simplification, for ψ is "harmonic" if and only if its component functions are solutions to the three-dimensional wave equation, and S is totally geodesic if its restriction to every straight line in $\mathbb{R}^n$ is of a particular explicit form—which we will describe later. Finally, for our purposes it suffices to take $n=3$ in order to obtain an angular momentum term, with a spherically symmetric exponential damping factor imposed.

2. DERIVATION OF THE SOLUTION

Let (ρ, ϕ, z) be cylindrical coordinates, that is,

$$x=\rho\cos\phi \qquad \rho^2=x^2+y^2$$
$$y=\rho\sin\phi \qquad d\phi=(-y\,dx+x\,dy)/\rho^2$$

Let (r, ϕ, θ) be spherical coordinates, that is,

$$\rho=r\sin\theta, \qquad z=r\cos\theta$$

The form of our metric is

$$ds^2=e^{m/r}(-dt^2+dx^2+dy^2+dz^2)$$
$$+2e^{m/r}\frac{k\rho^2}{r^3}\,d\phi\,(dt+dz)+e^{m/r}\frac{k^2\rho^2}{2r^6}(dt+dz)^2 \qquad (*)$$

Here m is the mass constant and k is the angular momentum constant.

In order to prove that $(*)$ is a solution to Rosen's field equations with flat background metric η [whose line element is $ds^2=-dt^2+dx^2+dy^2+dz^2$, and which η we regard as a diagonal matrix: $\eta=\mathrm{diag}(-1,1,1,1)$], we shall show that the matrix F of $(*)$ is a harmonic map

$$F:\mathbb{R}^4,\ \eta\to\mathcal{M},\ \mathcal{G}$$

where $\mathcal{M}$ is the manifold of symmetric 4×4 matrices of signature $(-,+,+,+)$, and $\mathcal{G}$ is the Dewitt metric (cf. Stoeger, 1983). One sees from $(*)$ that F is of the form

$$F=e^{m/r}\begin{bmatrix} -(1-\frac{1}{2}k^2\rho^2/r^6) & -ky/r^3 & kx/r^3 & \frac{1}{2}k^2\rho^2/r^6 \\ -ky/r^3 & 1 & 0 & -ky/r^3 \\ kx/r^3 & 0 & 1 & kx/r^3 \\ \frac{1}{2}k^2\rho^2/r^6 & -ky/r^3 & kx/r^3 & 1+\frac{1}{2}k^2\rho^2/r^6 \end{bmatrix}$$

In order to show that F is harmonic, it suffices to show that F is a composition

$$F = S \circ \psi$$

where $\psi: \mathbb{R}^4, \eta \to \mathbb{R}^3, |\circ|$ is a harmonic map, and where $S: \mathbb{R}^3, |\circ| \to \mathcal{M}, \mathcal{G}$ is totally geodesic, that is, where S maps straight lines in $\mathbb{R}^3$ into geodesics in $\mathcal{M}, \mathcal{G}$. Here, $|\circ|$ is the Euclidean norm (cf. Eells and Lemaire, 1980).

The easiest map to define is ψ. Our ψ is

$$\psi = \psi(t, x, y, z) = (m/r, ky/r^3, kx/r^3)$$

It is clear that ψ is harmonic, since each of the coordinate functions is harmonic.

The map S is defined as

$$S = S(\lambda, u, v) = e^{\lambda} \begin{bmatrix} -1+(u^2+v^2)/2 & -u & v & (u^2+v^2)/2 \\ -u & 1 & 0 & -u \\ v & 0 & 1 & v \\ (u^2+v^2)/2 & -u & v & 1+(u^2+v^2)/2 \end{bmatrix}$$

Lemma 1. $S = \eta \exp(\lambda I + uX + vY)$, where

$$I = \text{diag}\,(1, 1, 1, 1)$$

$$uX = \begin{bmatrix} 0 & +u & 0 & 0 \\ -u & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & -u & 0 & 0 \end{bmatrix}, \qquad vY = \begin{bmatrix} 0 & 0 & -v & 0 \\ 0 & 0 & 0 & 0 \\ v & 0 & 0 & v \\ 0 & 0 & v & 0 \end{bmatrix}$$

Proof. One readily checks that the matrices λI, uX, and vY all commute. Furthermore, we have that

$$(uX)(vY) = 0 = (vY)(uX)$$

$$(uX)^3 = (vY)^3 = 0$$

$$(vY)^2 + (uX)^2 = \begin{bmatrix} -(u^2+v^2) & 0 & 0 & -(u^2+v^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u^2+v^2 & 0 & 0 & u^2+v^2 \end{bmatrix}$$

Thus,

$$\begin{aligned}\exp\,(\lambda I+uX+vY)&\\ &=\exp(\lambda)\exp(uX)\exp(vY)\\ &=(\exp\lambda)[I+uX+\tfrac{1}{2}(uX)^2][I+vY+\tfrac{1}{2}(vY)^2]\\ &=(\exp\lambda)[I+uX+vY+\tfrac{1}{2}(uX)^2+\tfrac{1}{2}(vY)^2]\end{aligned}$$

From this it follows that $S=\eta\exp(\lambda I+uX+vY)$. ■

Lemma 2. Let $\lambda=ta+\lambda_0$, $u=tb+u_0$, $v=tc+v_0$ be the (geodesic) line in $\mathbb{R}^3$ through the point (λ_0, u_0, v_0) with direction numbers a, b, and c. Then

$$\sigma(t)=S(ta+\lambda_0,\, tb+u_0,\, tc+v_0)$$

is the geodesic through $S(\lambda_0, u_0, v_0)$ in the direction of the tangent vector represented by the symmetric matrix Z:

$$Z=\eta\exp(\lambda_0 I+u_0X+v_0Y)(aI+bX+cY) \qquad (**)$$

Proof. Differentiating $S(ta+\lambda_0, tb+u_0, tc+v_0)$ with respect to t yields $(**)$ at $t=0$. In particular, $(**)$ must be symmetric, since it is the derivative of a symmetric matrix varying with t.

Next, the geodesic through $S(\lambda_0, u_0, v_0)$ in the direction of Z is the function of t (Whitman *et al.*, 1986)

$$\sigma(t)=S(\lambda_0, u_0, v_0)\exp[tS(\lambda_0, u_0, v_0)^{-1}Z]$$

Hence the right side of $\sigma(t)$ is in fact

$$\begin{aligned}\sigma(t)&=\eta\exp(\lambda_0 I+u_0X+v_0Y)\exp[t(aI+bX+cY)]\\ &=\eta\exp[taI+\lambda_0 I+tbX+u_0X+tcY+v_0Y]\\ &=S(ta+\lambda_0, tb+u_0, tc+v_0)\quad ■\end{aligned}$$

Now Lemmas 1 and 2 yield the desired result:

Theorem. S is a totally geodesic map of $\mathbb{R}^3$, $|\circ|$ into $\mathcal{M}$. In particular, F is a harmonic map of $\mathbb{R}^4$ into $\mathcal{M}$.

This completes the proof that $(*)$ satisfies Rosen's field equations.

Note 1. A comparison of the metric $(*)$ with Rosen's static, spherically symmetric solution

$$ds^2=-e^{-m_1/r}\,dt^2+e^{m_2/r}(dx^2+dy^2+dz^2)$$

shows that taking $k=0$ yields the Rosen static solution with $m=-m_1=m_2$, i.e., it yields the conformally flat metric

$$ds^2=e^{m/r}(-dt^2+dx^2+dy^2+dz^2)$$

This metric (call it γ) is said to be "conformally flat," since the inclusion map

$$\begin{aligned}i\colon\quad &(R^4\backslash\mathbb{R}^1\times(0,0,0)),\ \gamma\to\mathbb{R}^4,\ \eta\\ &(t, x, y, z)\mapsto(t, x, y, z)\end{aligned}$$

preserves angles. The map i preserves neither geodesics nor geodesic trajectories, however. We look at the $e^{m/r}$ term as a spherically symmetric damping factor.

Note 2. If $k \neq 0$, there is an event horizon $r^2 = (k \sin \theta)/\sqrt{2}$ that is a "pinched" torus, a torus in which the center hole has zero diameter.

Note 3. The rank of the map $F: \mathbb{R}^4 \to \mathcal{M}$ is evidently three, in the sense that the differential of F has rank three as linear transformation.

3. PHYSICAL INTERPRETATION OF A STATIONARY AXISYMMETRIC SOLUTION TO ROSEN's FIELD EQUATIONS

This stationary axisymmetric solution to the field equations of N. Rosen does not seem to have a clear physical interpretation. It possesses the Kerr-like, time-independent $d\phi\, dt$ term, which can be interpreted as constant angular momentum in the z direction, due to rotation in the ϕ direction, around the z axis.

However, the $dt\, dz$ and $d\phi\, dz$ terms, which have the same constant of integration as the $d\phi\, dt$ term—i.e., the constant k—must be nonzero if $d\phi\, dt$ is. It might represent something like a time-independent momentum flux in the z direction (off the z axis), whose magnitude is closely linked to the angular momentum of the $d\phi\, dt$ term. It should be noted that, like the $dt\, d\phi$ term, the $d\phi\, dz$ and the $dt\, dz$ terms vanish on the z axis. It is difficult to see what this would mean.

For a Kerr-like axisymmetric solution we would like to keep the $d\phi\, dt$ term, allowing the $dt\, dz$ and $d\phi\, dz$ terms to vanish identically. But this cannot be done in equation (1). In fact, in the standard treatments of physically meaningful stationary axisymmetric solutions, one not only demands a metric form that is independent of ϕ (axisymmetry) and of t (stationarity), but also requires the symmetry $(\phi, t) \to (-\phi, -t)$ (cf. Matzner and Misner, 1967). This eliminates cross terms between (r, z) and (ϕ, t), for example, $d\phi\, dz$ and $dz\, dt$ terms.

However, in principle, we can conceive matter/momentum flows in the z direction that vanish on the z axis and would be the possible source of the metric given in equation (1). Such flow would emanate from the equatorial plane of the spinning source, moving parallel to, but not on, the z axis. The key problem with equation (1), as mentioned above, is the intimate link between the azimuthal flow and the flow in the z direction, due to the constant k. That does not render it physically meaningless, but rather a very peculiar solution in which any azimuthal flow necessitates flow in the z direction and vice versa. We are presently searching for other stationary axially symmetric solutions to Rosen's field equations that are more strictly analogous to Kerr.

REFERENCES

Courant, R., and Hilbert, D. (1962). *Methods of Mathematical Physics*, Interscience, New York.

Dewitt, B. S. (1967). Quantum theory of gravity, I, The canonical theory, *Physical Review*, **160**, 1113-1148.

Eells, J., and Lemaire, L. (1978). A report on harmonic maps, *Bulletin of the London Mathematical Society*, **10**, 1-68.

Eells, J., and Lemaire, L. (1980). *Selected Topics in Harmonic Maps*, Tulane, CBMS Regional Conference Series #50.

Eells, J., and Sampson, J. H. (1964). Harmonic mappings of Riemannian manifolds, *American Journal of Mathematics* **86**, 109-160.

Fuller, F. B. (1954). Harmonic mappings, *Proceedings of the National Academy of Science of the USA*, **40**, 987-991.

Knill, R., Kalka, M., and Sealey, H. (1982). *Proceedings of the Tulane Harmonic Maps Conference*, Springer-Verlag Lecture Notes in Mathematics #949, New York.

Kobayaski, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Wiley, New York.

Matzner, R. A., and Misner, C. W. (1967). Gravitational field equations for sources with axial symmetry and angular momentum, *Physical Review*, **154**, 1229-1232.

Misner, C. W. (1978). Harmonic maps as models for physical theories, *Physical Review D*, **18**, 4510-4523.

Rosen, N. (1973). A Bi-metric theory of gravitation, *General Relativity and Gravitation* **4**, 435-447.

Rosen, N. (1974). A theory of gravitation, *Annals of Physics*, **84**, 455-473.

Rosen, N. (1975). A Bi-metric theory of gravitation, II, *General Relativity and Gravitation*, **6**, 259-268.

Rosen, N. (1980). General Relativity with a background metric, *Foundations of Physics*, **10**, 673-704.

Sampson, J. H. (1978). Some properties and applications of harmonic mappings, *Annales Scientifiques de l'Ecole Normale Supérieure*, **11**, 211-228.

Stoeger, W. R. (1983). Rosen's bi-metric theory of gravity as a harmonic map, in *Proceedings of the third Marcel Grossmann Meeting on Recent Developments in General Relativity*, Hu Ning, ed. Part B, pp. 921-925, North-Holland, Amsterdam.

Stoeger, W. R. (1978). Orbital topography and other astrophysical consequences of Rosen's bi-metric theory of gravity, *General Relativity and Gravitation* **9**, 165-174.

Stoeger, W. R. (1979). Axisymmetric accretion flows very near black holes and Rosen-collapsed objects, *General Relativity and Gravitation* **10**, 671-679.

Stoeger, W. R. (1980). Rapid variability, dying pulse train, and black holes, *Monthy Notices of the Royal Astronomical Society* **190**, 715-772.

Whitman, A., Knill, R., and Stoeger, W. (1986). Some harmonic maps on pseudo-Riemannian manifolds, *International Journal of Theoretical Physics*, **25**, 1139-1153.